Iterative structures in $\mathcal{N}=4$ super Yang-Mills and $\mathcal{N}=8$ supergravity

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based on work with
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Supersymmetry $\Rightarrow$ improved UV behavior

$\mathcal{N}=4$ SYM

- UV finite
- $\beta = 0$ $\Rightarrow$ superconformal
- $SU(N)$ theory at large N and strong coupling dual to IIB / supergravity on $AdS_5 \times S^5$

interpolate?

$\mathcal{N}=8$ supergravity

- pure gravity diverges at 2 loops
- supergravity diverges at ≥ 3 loops
- $\mathcal{N}=8$ diverges at >3 loops : finite?

Despite good UV behavior,
gauge and gravity theories
are perturbatively complicated,
in terms of Feynman diagrams

Nevertheless, the tree- and loop-
amplitudes are often much
simpler than one would expect
esp. in maximally supersymmetric
theories

1980's
- Gauge theory tree amplitudes — Parke, Taylor
- $\mathcal{N}=4$ loop amplitudes — Green, Schwarz, Brink; Bern, Kosower
- Gravity theory — Kawai, Lewellen, Tye

suggesting that alternative
calculational approaches may
be called for.

In the 1990's, Bern, Dixon, Kosower and collaborators developed unitarity-based methods for calculating loop integrals in gauge and gravity theories

$\Rightarrow$ 3, 4, 5 - loop calculations

In 2003-5, iterative relations were discovered for leading-color $\mathcal{N}=4$ SYM L-loop amplitudes in terms of lower loops $\Rightarrow$ BDS ansatz

$$A \sim \exp\left(A^{1\text{-loop}}\right)$$

Today:

1) iterative relations for $\mathcal{N}=8$ supergravity amplitudes

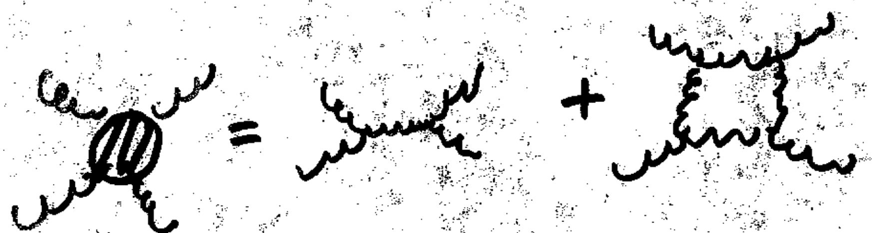
2) relations for subleading-color $\mathcal{N}=4$ SYM amplitudes

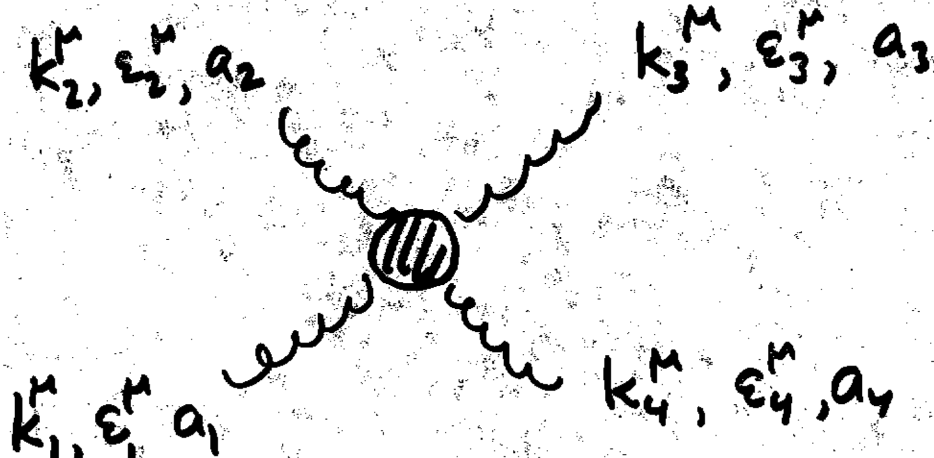
3) Link between $\mathcal{N}=8$ supergravity amp. and most-subleading-color $\mathcal{N}=4$ SYM amp. at loop level

$\mathcal{N}=4$ SYM four-gluon amplitudes

Loop expansion:

$$\mathcal{A} = g^2 \mathcal{A}^{(0)} + g^4 \mathcal{A}^{(1)} + \ldots = g^2 \sum_{L=0}^{\infty} g^{2L} \mathcal{A}^{(L)}$$





$$\mathcal{A}^{(L)}(k_i^\mu, \varepsilon_i^\mu, a_i)$$

- color
- polarization (helicity)
- momenta

Color decomposition and $\frac{1}{N}$ expansion:

At tree level

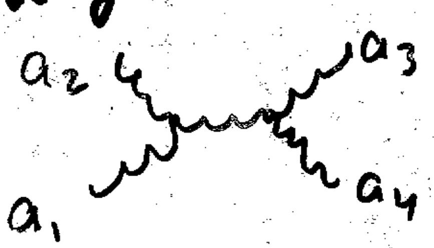
$$\mathcal{A}^{(0)}(k_i\, \varepsilon_i\, a_i)$$

"color-ordered amplitudes"

$$= \sum_{\sigma \in S_3} A^{(0)}(\sigma(k,\varepsilon))\, \mathrm{Tr}\left[T^{a_1} T^{\sigma(a_2)} T^{\sigma(a_3)} T^{\sigma(a_4)}\right]$$

(T^a = generators in fundamental rep.)

Why T^a?



$$f^{a_1 a_2 b} f^{a_3 a_4 b} \sim \mathrm{Tr}\left([T^{a_1}, T^{a_2}]T^b\right) \mathrm{Tr}\left([T^{a_3}, T^{a_4}]T^b\right)$$

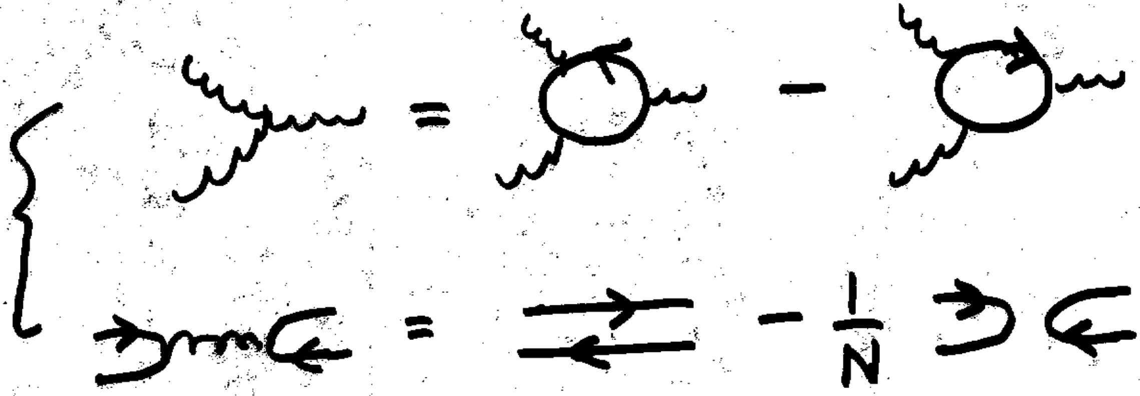
$$\sim \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - (a_1 \leftrightarrow a_2)$$

$$-(a_3 \leftrightarrow a_4) + \begin{pmatrix} a_1 \leftrightarrow a_2 \\ a_3 \leftrightarrow a_4 \end{pmatrix}$$

other diagrams give
2 more four-trace terms

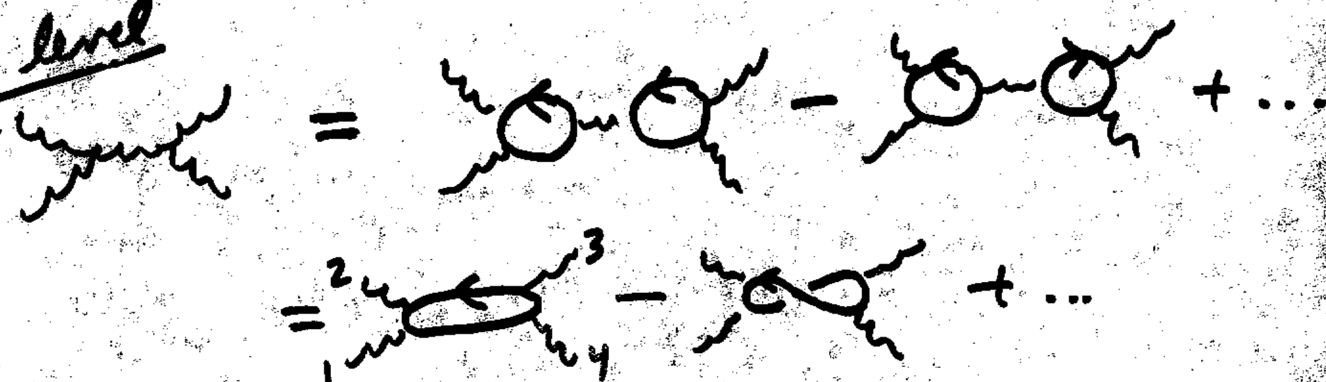
Graphical (double line) notation
't Hooft

Rules



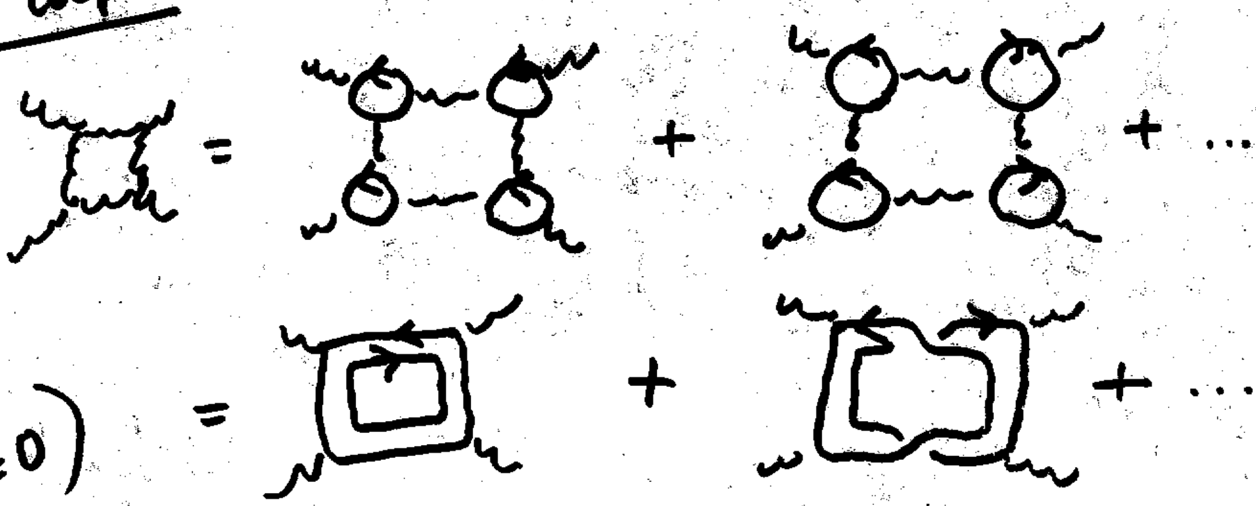
$-\frac{1}{N}$

Tree level



$$= g^2\left[\,\mathrm{Tr}(1234) - \mathrm{Tr}(1243) - \mathrm{Tr}(2134) + \mathrm{Tr}(2143)\,\right]$$

one loop



$(\mathrm{Tr}(T^a)=0)$

$$= g^2\Big[\, \underbrace{g^2 N\, \mathrm{Tr}(1234)}_{\text{single-trace}} + \underbrace{g^2\, \mathrm{Tr}(12)\,\mathrm{Tr}(34)}_{\text{double-trace}} + \dots \Big]$$

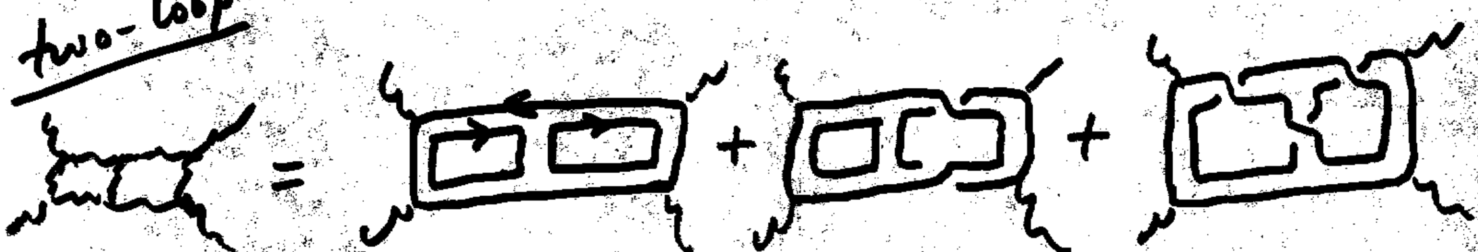
$$\mathcal{A}^{(1)} = g^2 \Big[\sum_{\sigma \in \frac{S_4}{\mathbb{Z}_4}} \underbrace{g^2 N}_{\lambda} A^{(1,0)}(\sigma(k,\epsilon)) \, \mathrm{Tr}\big(\sigma(1)\sigma(2)\sigma(3)\sigma(4)\big)$$

leading-color

$$+ \sum_{\sigma \in \frac{S_4}{\mathbb{Z}_2^2}} \underbrace{g^2}_{\frac{\lambda}{N}} A^{(1,1)}(\sigma(k,\epsilon)) \, \mathrm{Tr}\big(\sigma(1)\sigma(2)\big) \, \mathrm{Tr}\big(\sigma(3)\sigma(4)\big) \Big]$$

subleading-color

<u>two-loop</u>



$$= g^2 \Big[g^4 N^2 \mathrm{Tr}(1234) + g^4 N \, \mathrm{Tr}(12)\,\mathrm{Tr}(34) + g^4 \, \mathrm{Tr}(1234) \Big]$$

leading-color single-trace | double trace | subleading-color single trace

$$\mathcal{A}^{(2)} = g^2 \Big[\sum_{\frac{S_4}{\mathbb{Z}_4}} \Big(\lambda^2 A^{(2,0)} + \frac{\lambda^2}{N^2} A^{(2,2)} \Big) \mathrm{Tr}\big(\sigma(1)\sigma(2)\sigma(3)\sigma(4)\big)$$

$$+ \sum_{\frac{S_4}{\mathbb{Z}_2^2}} \frac{\lambda^2}{N} A^{(2,1)} \, \mathrm{Tr}\big(\sigma(1)\sigma(2)\big) \, \mathrm{Tr}\big(\sigma(3)\sigma(4)\big) \Big]$$

L-loops

$$\mathcal{A}^{(L)} = g^2 \Big[\sum_{S_4/Z_4} \lambda^L \sum_k \frac{1}{N^{2k}} A^{(L,2k)} \mathrm{Tr}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))$$

$$+ \sum_{S_4/Z_2^2} \lambda^L \sum_k \frac{1}{N^{2k+1}} A^{(L,2k+1)} \mathrm{Tr}(\sigma(1)\sigma(2))\,\mathrm{Tr}(\sigma(3)\sigma(4)) \Big]$$

$$\underbrace{A^{(L,0)}}_{\text{leading-color}},\ \underbrace{A^{(L,1)}, \ldots, A^{(L,L)}}_{\text{subleading-color}}$$

Leading-color $(N \to \infty)$, "planar" approximation

$$\mathcal{A}^{(L)} = g^2 \sum_{S_4/Z_4} \lambda^L A^{(L,0)} \mathrm{Tr}(\sigma(1)\sigma(2)\sigma(3)\sigma(4))$$

$$\mathcal{A}(k_i^\mu, \varepsilon_i^\mu, a_i) \begin{cases} \text{color } \checkmark \\ \text{helicity} \\ \text{momenta} \end{cases} = A(k_i, \varepsilon_i)\,\mathrm{Tr}(1234) + \ldots$$

$$A^{(0)}(k_i, \varepsilon_i) = -\frac{4iK}{st} \leftarrow$$ valid for supersymmetric $\underline{\text{or}}$ non supersymmetric YM theory (tree level)

$$\begin{cases} s = (k_1+k_2)^2 \\ t = (k_1+k_4)^2 \\ u = (k_1+k_3)^2 \end{cases}$$

momentum conservation

$$\Rightarrow \quad s+t+u=0$$

$K(k_i, \varepsilon_i) =$ invariant under all permutations of external legs

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$$\left.\begin{array}{l} A^{(0)} = 0 \text{ for helicities } ++++ \\ \qquad\qquad\qquad\quad \text{ or } +++- \\ A^{(0)} \neq 0 \text{ for } ++--,\ +-+- \end{array}\right\}$$

This remains true at all loops for supersymmetric theories

For $\mathcal{N}=4$ SYM, the helicity dependence of all loop amplitudes is also given by K, thus

$$A^{(L)}(k_i, \varepsilon_i) = K \cdot \begin{pmatrix} \text{momentum} \\ \text{dependent} \\ \text{function} \end{pmatrix}$$

Helicity decomposition

For $\mathcal{N}=4$ SYM:

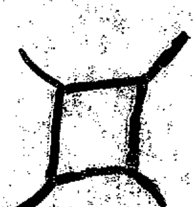
$$A^{(L,0)} = A^{(0)} \, M^{(L)}(k_i)$$

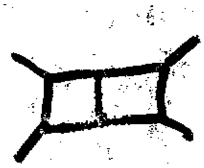
$A^{(L,0)}$: leading-color, MHV L-loop amplitude

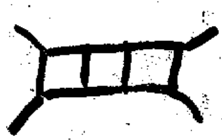
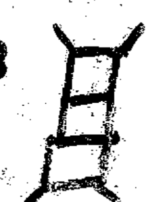
$A^{(0)}$: tree amplitude

$M^{(L)}$: only depends on momenta (s,t)

Bern, Dixon, Kosower + collaborators showed that $M^{(L)}$ are expressible in terms of scalar loop integrals (planar)

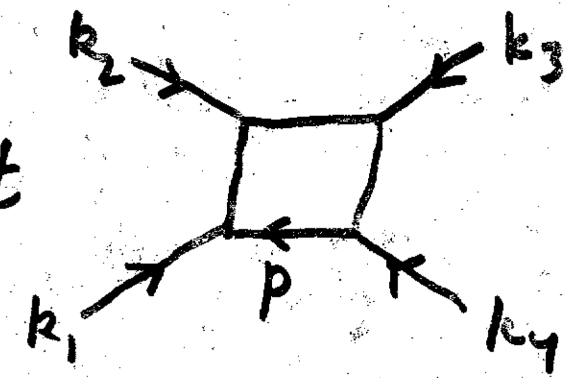
$M^{(1)} \sim st$ 

$M^{(2)} \sim s^2 t$  $+ st^2$ 

$M^{(3)} \sim s^3 t$  $+ st^3$ 

$+ st^2$  $+ \ldots$

Loop integrals

$M^{(1)} \sim \quad s\,t$ 

$$\sim st \int \frac{d^4 p}{p^2 (p+k_1)^2 (p+k_1+k_2)^2 (p-k_4)^2}$$

Diverges, not in the UV ($N=4$ is finite)
but in the IR, due to
masslessness of the gluons

Use dimensional regularization by
doing the integrals in $D = 4-2\epsilon$ dimensions

Divergences appear as poles in ϵ.

(In physical quantities, there poles
cancel against amplitudes for
emission of arbitrary soft gluons.)

one-loop amplitude

$$M^{(1)} \sim \frac{2}{\epsilon^2}\left(\frac{\mu^2}{-s}\right)^{\epsilon} + \frac{2}{\epsilon^2}\left(\frac{\mu^2}{-t}\right)^{\epsilon} - \log^2\left(\frac{s}{t}\right) - \frac{4\pi^2}{3}$$
$$+ \epsilon\left(\dots \mathrm{Li}_3\left(-\frac{t}{s}\right) + \dots\right) + \dots$$

$\frac{1}{\epsilon}$ each for soft + collinear divergences

two-loop amplitude

$M^{(2)} \sim s^2 t$  $+ \; s t^2$ 

$$\sim \frac{8}{\epsilon^4} + \dots + \left(\dots \mathrm{Li}_4\left(-\frac{t}{s}\right) + \dots\right) + O(\epsilon)$$

$\frac{1}{\epsilon^2}$ for each loop

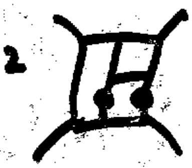
In 2003, Anastasiou, Bern, Dixon + Kosower observed that

$$\underset{O\left(\frac{1}{\epsilon^4}\right)}{M^{(2)}} = \frac{1}{2}\underbrace{\left[M^{(1)}\right]^2}_{O\left(\frac{1}{\epsilon^4}\right)} - (\zeta_2 + \epsilon\zeta_3 + \epsilon^2\zeta_4)\underbrace{M^{(1)}(2\epsilon)}_{O\left(\frac{1}{\epsilon^2}\right)} - \frac{\pi^2}{72} + O(\epsilon)$$

$M^{(2)}$ determined by $M^{(1)}$ and several constants: ζ_2, ζ_3, $\frac{\pi^2}{72}$

(ABDK relation)

three-loop amplitude

$M^{(3)} \sim s^3 t$  $+ s t^3$  $+ s t^2$  $+ \dots$

$$\sim \frac{32}{3} \frac{1}{\epsilon^6} + \dots$$

$$M^{(3)} = M^{(1)} M^{(2)} - \frac{1}{3}[M^{(1)}]^3 + \left(\frac{11}{2}\zeta_4 + \left(6\zeta_5 + 5\zeta_2\zeta_3\right)\epsilon \right) M^{(1)}(3\epsilon) + C_3 + O(\epsilon)$$

$M^{(3)}$ determined by $M^{(1)}$ and a few more constants; thus

$$\begin{aligned} A &= \sum \lambda^L A^{(L,0)} \\ &= A^{(0)} \sum \lambda^L M^{(L)} \\ &= A^{(0)} \Big[1 + \lambda M^{(1)} \\ &\quad + \lambda^2 \left(\tfrac{1}{2} M^{(1)2} - [\zeta_2 + \epsilon \zeta_3] M^{(1)}(2\epsilon) + C_2 \right) \\ &\quad + \lambda^3 \left(M^{(1)} M^{(2)} - \tfrac{1}{3} M^{(1)3} + [\tfrac{11}{2}\zeta_4 + \dots] M^{(1)}(3\epsilon) \right. \\ &\quad + \dots \\ &= A^{(0)} \exp \Big[\lambda M^{(1)}(\epsilon) \\ &\quad + \lambda^2 (\zeta_2 + \epsilon \zeta_3) M^{(1)}(2\epsilon) + C_2 \\ &\quad + \lambda^3 \left(\tfrac{11}{2}\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) \right) M^{(1)}(3\epsilon) + C_3 \\ &\quad + \dots \Big] \end{aligned}$$

BDS all-loop ansatz

$$A = A^{(0)} \exp\Big[\sum_{L=1}^{\infty} \lambda^L \left(f^{(L)} + \epsilon g^{(L)} \right) M^{(1)}(L\epsilon) + C_L \Big]$$

where all the constants $f^{(L)}$ and $g^{(L)}$ can be formed into 2 functions of λ

$$f(\lambda) = \lambda - \zeta_2 \lambda^2 + \frac{11}{2}\zeta_4 \lambda^3 + \dots$$
$$g(\lambda) = -\zeta_3 \lambda^2 + (6\zeta_5 + 5\zeta_2\zeta_3)\lambda^3 + \dots$$

known as the cusp anomalous dimension and collinear anomalous dimension

(A also depends on the constants C_L.)

The BDS ansatz for 4-gluon amplitudes
(supported by strong coupling
Alday-Maldacena calculation using adS/CFT

and the "dual conformal symmetry"
argument of
Drummond, Korchemsky, and Sokatchev)
shows an iterative, exponential structure
for leading-color $N=4$ SYM amplitudes

<u>Remainder of talk</u>

I. Analogous behavior for
$N=8$ supergravity amplitudes

II. ... and subleading-color
$N=4$ SYM amplitudes
$A^{(L,k)}$

III. Link between supergravity amplitudes $A^{(L)}_{SG}$
and most-subleading $N=4$ ampl. $A^{(L,L)}_{SYM}$

I. $\mathcal{N}=8$ supergravity amplitudes

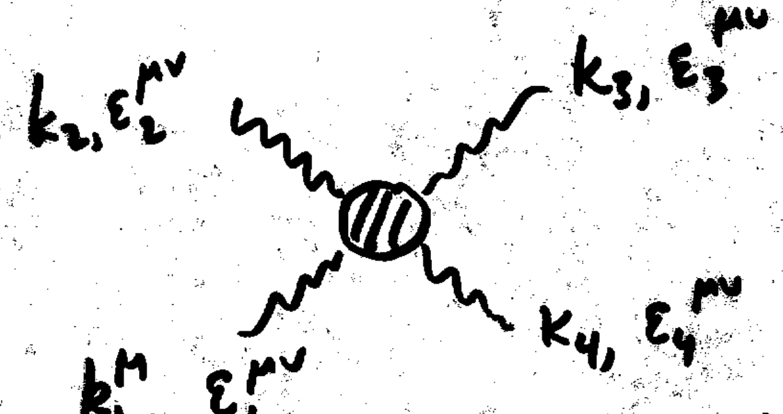
Loop expansion of 4-graviton amplitude:

$$\mathcal{A} = \kappa^2\left[\mathcal{A}^{(0)} + \kappa^2\mathcal{A}^{(1)} + \kappa^4\mathcal{A}^{(2)} + \kappa^6\mathcal{A}^{(3)} + \dots\right]$$

UV finite at least to here

$\kappa^2 \sim G_N \sim \frac{1}{M^2_{planck}}$

(no color indices)



$\varepsilon_i^{\mu\nu} = \varepsilon_i^{\mu}\varepsilon_i^{\nu}$

Here, perturbation theory is much worse, but answers are as "simple" as for $\mathcal{N}=4$ SYM.

Tree-level

$$\mathcal{A}^{(0)} = \frac{16 i \kappa^2}{stu}$$

$$\Rightarrow \quad stu\,\mathcal{A}^{(0)}_{SG} = -i\left[st\,A^{(0)}_{SYM}\right]^2$$

follows from Kawai, Lellewen, Tye string theory relation

closed string $\sim$ (open string$)^2$

For $N=8$ supergravity:

$$A^{(L)} = A^{(0)} M^{(L)}$$

where, as before, $M^{(L)}$ can be expressed in terms of scalar loop diagrams

one-loop amplitude:

$$M^{(1)} = stu\left(I(s,t) + I(t,u) + I(u,s)\right)$$

$I(s,t) \sim$  $\sim \dfrac{4}{st\epsilon^2} + \cdots$

Thus

$$M^{(1)} = \frac{4(u+s+t)}{\epsilon^2} + \frac{1}{\epsilon}(\cdots)$$

vanishes by momentum conservation and masslessness of the graviton

1-loop gravity has $O(\frac{1}{\epsilon})$ IR divergence (soft, but no collinear divergence)

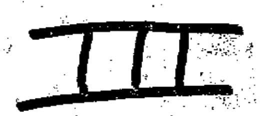
$$M^{(1)} \sim \frac{1}{\epsilon}\left[s\log(-s) + t\log(-t) + u\log(-u)\right]$$
$$+ s\log(-t)\log(-u) + t\log(-u)\log(-s) + u\log(-s)\log(-t)\Big]$$
$$+ O(\epsilon^2)$$

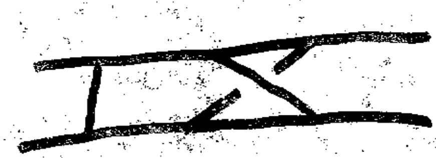
Subtlety: analytic continuation required since $s+t+u=0$.

two-loop-amplitude

$$M^{(2)} = s^3 t u \left(I^{(2P)}(s,t) + I^{(2NP)}(s,t) \right)$$

+ permutations

$I^{(2P)} \sim$  (planar)

$I^{(2NP)} \sim$  (nonplanar)

 $\sim O\left(\frac{1}{\epsilon^4}\right)$

 $\sim O\left(\frac{1}{\epsilon^4}\right)$

but $\frac{1}{\epsilon^4}$ and $\frac{1}{\epsilon^3}$ <u>cancel</u> in $M^{(2)}$

$$M^{(2)} \sim \frac{1}{2\epsilon^2}\left[s\log(-s) + t\log(-t) + u\log(-u) \right]^2 + \dots$$

In fact

$$M^{(2)} = \frac{1}{2}\left[M^{(1)}\right]^2 + O(\epsilon^0) \quad ! \quad \left(\text{analogous to ABDK}\right)$$

S.N, Nastase, Schnitzer 0805.2347

Brandhuber, Heslop, Nasti, Spence, Travaglini: 0805.2763

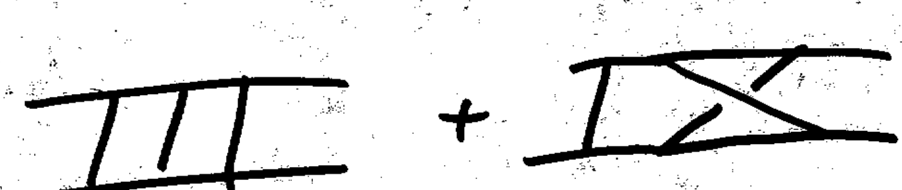
The $O(\epsilon^0)$ piece is not constant as in the ABDK relation, but it is "relatively simple"

$$su\Big[\,8S_{1,3}\left(-\tfrac{s}{t}\right)+\tfrac{1}{3}\log^4\left(-\tfrac{s}{t}\right)+8\zeta_4$$

$$+i\pi\left(-8S_{12}\left(-\tfrac{s}{t}\right)+\tfrac{4}{3}\log^3\left(-\tfrac{s}{t}\right)+8\zeta_3\right)\Big]$$

+ permutations.

Iterative relation unexpected because $M^{(2)}$ involves nonplanar diagrams, but in fact they are necessary for $M^{(2)}\sim\frac{1}{2}[M^{(1)}]^2$ to work



Does this continue to higher loops?

$$M^{(L)}\sim\frac{1}{L!}[M^{(1)}]^L$$

$\Rightarrow M\sim\exp[M^{(1)}]$. maybe!

(3-loop nonplanar integrals not yet computed.)

II. Iterative relations for subleading-color $\mathcal{N}=4$ SYM amplitudes?

$A^{(L,k)}$ $(k=1,\ldots,L)$

These involve non-planar diagrams, but so does $\mathcal{N}=8$ supergravity

Recall that leading-color amplitudes have IR poles:

$$A^{(L,0)} \sim \frac{1}{\epsilon^{2L}} + \ldots$$

one-loop subleading amplitude

$$A^{(1,1)} \sim K\left(I(s,t) + I(t,u) + I(u,s) \right)$$

$$\frac{4}{st\epsilon^2} + \frac{4}{tu\epsilon^2} + \frac{4}{us\epsilon^2}$$

$$= \frac{4}{stu\,\epsilon^2}(s+t+u) = 0$$

So

$$A^{(1,1)} \sim \frac{1}{\epsilon}\left(s\log(-s) + t\log(-t) + u\log(-u) \right)$$

$$+ \ldots$$

Two-loop subleading color amplitudes

$$A^{(2,1)} \sim K\Big[\, s\big(3I^{(2P)}(s,t) + 2I^{(2NP)}(s,t) + 3I^{(2P)}(s,u) + 2I^{(2NP)}(s,u)\big) - t\big(I^{(2NP)}(t,s) + I^{(2NP)}(t,u)\big) - u\big(I^{(2NP)}(u,s) + I^{(2NP)}(u,t)\big)\Big]$$

$I^{(2P)} \sim$  $\sim \frac{1}{\epsilon^4}$, $I^{(2NP)} \sim$  $\sim \frac{1}{\epsilon^4}$

$$A^{(2,1)} \sim O\left(\frac{1}{\epsilon^3}\right)$$

$$A^{(2,2)} \sim K\Big[\, s\big(I^{(2P)}(s,t) + I^{(2NP)}(s,t) + (t \leftrightarrow u)\big) + t\big(I^{(2P)}(t,s) + I^{(2NP)}(t,s) + (s \leftrightarrow u)\big) - 2u\big(I^{(2P)}(u,s) + I^{(2NP)}(u,s) + (s \leftrightarrow t)\big)\Big]$$

$$A^{(2,2)} \sim O\left(\frac{1}{\epsilon^2}\right)$$

In general, we found

$$A^{(L,k)} \sim O\left(\frac{1}{\epsilon^{2L-k}}\right)$$

based on IR formalism of Catani, and Sterman and Tejeda-Yeomans

We computed in 0809.0376 the coefficients of the leading poles of $A^{(L,k)}$.

In particular, we found

$$A^{(L,L)}(\epsilon) \sim \frac{P_{L-1}(x,y,z)}{\epsilon^{L-1}} A^{(1,1)}(L\epsilon) + O\left(\frac{1}{\epsilon^{L-2}}\right)$$

where $P_n(x,y,z)$ is a known n^{th} order polynomial of

$$x = \log\left(\frac{t}{u}\right)$$
$$y = \log\left(\frac{u}{s}\right)$$
$$z = \log\left(\frac{s}{t}\right)$$

an iterative relation between the <u>most-subleading</u> L-loop amplitude & the most-subleading 1-loop amplitude.

(only holds for the 1^{st} 2 poles)

III. Link between

$N=8$ supergravity amplitudes

$$A_{SG}^{(L)} = A_{SG}^{(0)} M_{SG}^{(L)}$$

and

most-subleading-color $N=4$ SYM amplitudes

$$A_{SYM}^{(L,L)} = A_{SYM}^{(0)} M_{SYM}^{(L,L)}$$

- Both have IR divergences starting at $O\left(\frac{1}{\epsilon^L}\right)$

- $$M_{SG}^{(1)} \sim u\, M_{SYM}^{(1,1)}(s,t)$$

$$\left(\sim I(s,t) + I(s,u) + I(t,u)\right)$$

- By studying IR divergences, we found

$$M_{SG}^{(2)} \sim u^2 M_{SYM}^{(2,2)}(s,t) + \left(\begin{matrix}\text{cyclic} \\ \text{perms.}\end{matrix}\right)$$

which is exact!

Could this generalize?

$$M_{SG}^{(L)} \overset{?}{\sim} u^L M_{sym}^{(L,L)}(s,t) + \begin{pmatrix} \text{cyclic} \\ \text{perm} \end{pmatrix}$$

- has correct dimensionality
- implies finiteness of $N=8$ supergravity!

- $M_{SG}^{(L)}$ not calculated for $L>2$
- Leading divergence of $M_{SG}^{(L)}(\epsilon)$ expected to be $\sim \frac{1}{L!}\left[M_{SG}^{(1)}\right]^2$
- Leading divergence of $M_{sym}^{(L,L)}(\epsilon)$ given by $\sim \dfrac{P_{L-1}(x,y,z)\, M^{(1,1)}(L\epsilon)}{\epsilon^{L-1}}$

- They <u>don't</u> match.
- Is this just a "2-loop accident" or can it be corrected?

Conclusions

- evidence for iterative structure for $\mathcal{N}=8$ supergravity for the IR divergent terms

$$M_{SG}^{(L)} = \frac{1}{L!}\left[M_{SG}^{(1)}\right]^{L} + \text{finite}$$

- evidence for iterative structure for $\mathcal{N}=4$ subleading-color amplitudes

$$M_{sym}^{(L,L)} = \frac{P_{L-1}(x,y,z)}{\epsilon^{L-1}} M_{sym}^{(1,1)}(L\epsilon) + \ldots$$

- evidence for loop-level relation between $N=8$ supergravity and $N=4$ SYM

$$M_{SG}^{(L)} \sim u^{L} M_{sym}^{(1,1)}(s,t) + \text{cyclic}$$

at least for $L \leq 2$

Thanks!